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LETTER TO THE EDITOR

Bi-Hamiltonian structure of an integrable Hénon–Heiles system

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Abstract. By making use of the system of coordinates in which the separation of the variables in the Hamilton–Jacobi equation takes place, we find a bi-Hamiltonian structure of a two degrees of freedom Hamiltonian system corresponding to the integrable Hénon–Heiles Hamiltonian $H=\frac{1}{2}(p_q^2+p_2^2+Aq_1^2+Bq_2^2)-q_1^2q_2-2q_2^3$.

Let there be given on $\mathbb{R}^n$ a Hamiltonian vector field v for the Poisson structure (Poisson bracket) V. It means that the flow of v preserves V, or equivalently, there exists a smooth Hamiltonian function H_V such that $v=V[\cdot, H_V]$ (see [2]). Suppose now that v preserves a second Poisson structure W. If V and W are everywhere non-degenerate and compatible (i.e. all of their linear combinations $\lambda V+\mu W$ are also Poisson structures) then the following theorem holds.

Theorem (Dorfman and Gel'fand [2, 4]). There exists a sequence of smooth functions $\{f_k\}$, such that:

(a) f_1 is a Hamiltonian of the vector field v with respect to V;

(b) the vector field of the V Hamiltonian f_k coincides with the vector field of the W Hamiltonian f_{k+1};

(c) the functions $\{f_k\}$ are in involution with respect to both Poisson brackets.

The importance of the above theorem lies in the fact that in certain cases the arising series $\{f_k\}$ of functions in involution provides for the complete integrability of the vector field v.

To our knowledge, however, the known examples of finite-dimensional vector fields satisfying the conditions of the above theorem fall within two groups:

(i) systems with Casimir functions, i.e. the brackets V and W are degenerate (such as the Toda lattice [2]);

(ii) systems which can be presented as a direct product of two other Hamiltonian systems (the existence of a second Poisson structure is trivial).

Definition. The Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}f=V[f, H_v] \tag{1}$$

possesses a bi-Hamiltonian structure provided that there exist a function $\rho \not\equiv 0$ and a Poisson structure W such that:

(i) the flow of the vector field $\rho V[\cdot, H_V]$ preserves W;

(ii) the corresponding first integrals H_V and H_W are functionally independent.

If (1) is a two degrees of freedom system having a bi-Hamiltonian structure then it is completely integrable. The second integral H_W is easily reconstructed from the Poisson structures V, W and the functions ρ, H_V. If $N > 2$ then it is an open question whether the Dorfman–Gel'fand theorem can be generalized for systems having a bi-Hamiltonian structure with $\rho \not\equiv$ constant.

In the present letter we shall find a bi-Hamiltonian structure (with $\rho \not\equiv$ constant) of a two degree of freedom Hamiltonian system (1) which does not fall within the two groups of examples described above.

Let H be the following integrable Hénon–Heiles Hamiltonian [3]:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2) - q_1^2 q_2 - 2q_2^3 \tag{2}$$

and V be the standard Poisson structure on $\mathbb{R}^4\{p_1, p_2, q_1, q_2\}$

$$V = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2}.$$

The second integral of the system (1) reads

$$F = q_1^4 + 4q_1^2 q_2^2 + 4p_1(p_1 q_2 - p_2 q_1) - 4Aq_1^2 q_2 + (4A - B)(p_1^2 + Aq_1^2).$$

We shall use the following (u, v) variables which are known to be separable for the corresponding Hamilton–Jacobi equation [1, 5]

$$q_1^2 = -4uv \qquad q_2 = u + v + (B - 4A)/4. \tag{3}$$

The canonical variables (p_u, p_v, u, v) on $T^*\mathbb{R}^2$ are given by

$$p_1 = \frac{(-uv)^{1/2}(p_u - p_v)}{u - v} \qquad p_2 = \frac{up_u - vp_v}{u - v}. \tag{4}$$

In these new variables the integrals of motion take the form

$$\begin{aligned} H = \{&4up_u^2 - 4vp_v^2 - 16(u^4 - v^4) + 8(6A - B)(u^3 - v^3) + (B - 4A)(12A - B)(u^2 - v^2) \\ &+ A(4A - B)^2(u - v)\}/8(u - v) \end{aligned}$$

$$\begin{aligned} F = uv\{&4p_u^2 - 4p_v^2 - 16(u^3 - v^3) + 8(6A - B)(u^2 - v^2) \\ &+ (B - 4A)(12A - B)(u - v)\}/(u - v) \end{aligned}$$

and

$$V = \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial p_u} + \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial p_v}.$$

A direct computation gives

$$\begin{aligned} \frac{\partial F}{\partial u} &= 8v \frac{\partial H}{\partial u} \qquad & \frac{\partial F}{\partial v} &= 8u \frac{\partial H}{\partial v} \\ \frac{\partial F}{\partial p_u} &= 8v \frac{\partial H}{\partial p_u} \qquad & \frac{\partial F}{\partial p_v} &= 8u \frac{\partial H}{\partial p_v}. \end{aligned} \tag{5}$$

The identities (5) suggest to define a new Poisson structure

$$W = u \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial p_u} + v \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial p_v}.$$

Proposition. (V, W) provides a bi-Hamiltonian structure of the Hamiltonian system (1) where H is the integrable Hénon–Heiles Hamiltonian (2).

Proof. One trivially verifies that W satisfies the Jacobi identity and hence W is a Poisson structure. Using (5) we obtain $V[\cdot, H] = \rho W[\cdot, F]$, where $\rho = 8uv = -2q_1^2$. □

In the initial coordinates (p_1, p_2, q_1, q_2) the second Poisson structure W takes the form

$$W = \tfrac{1}{2}q_1 \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_2} + \tfrac{1}{2}q_1 \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_1} + \left(q_2 + \frac{4A-B}{4}\right)\frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2} - \tfrac{1}{2}p_1 \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2}. \quad (6)$$

Let us define now a third Poisson structure

$$U = \frac{1}{u}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial p_u} + \frac{1}{v}\frac{\partial}{\partial v} \wedge \frac{\partial}{\partial p_v}.$$

The same arguments as above show that (U, V) provides a bi-Hamiltonian structure of the Hamiltonian vector field $V[\cdot, F]$, as $V[\cdot, F] = \rho U[\cdot, H]$, where $\rho = 8uv = -2q_1^2$. In the initial coordinates (p_1, p_2, q_1, q_2) the Poisson structure U reads

$$U = \frac{B-4A+4q_2}{q_1^2}\frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{2}{q_1}\frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_2} + \frac{2}{q_1}\frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_1} - \frac{2p_1}{q_1^2}\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2}.$$

Finally we note that the three Poisson structures U, V, W, turn out to be compatible each to the other.

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